RHEOLOGICAL MODELS

Lecture 11

PFE-2.4.5

By:

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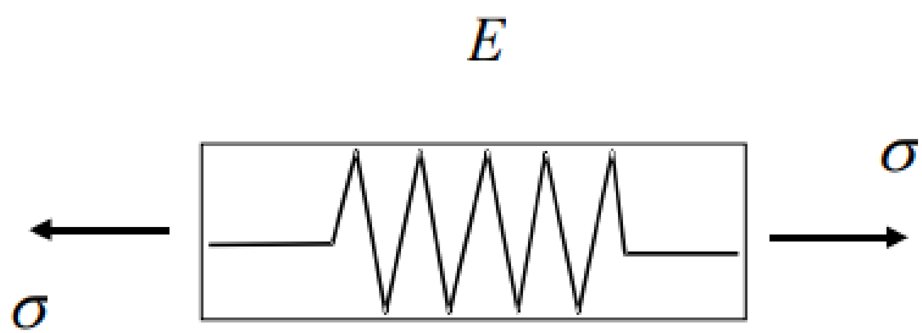
The mechanical models

- **Viscoelastic**: "viscous" + "elastic"; a viscoelastic material exhibits both viscous and elastic behaviour – a bit like a fluid and a bit like a solid.
- To get some feeling for linear viscoelastic behaviour, it is useful to consider the simpler behaviour of analog mechanical models.
- They are constructed from linear springs and dashpots, disposed singly and in branches of two (in series or in parallel)

The Linear Elastic Spring

The response of this material to a creep-recovery test is to undergo an instantaneous elastic strain upon loading, to maintain that strain so long as the load is applied, and then to undergo an instantaneous de-straining upon removal of the load.



$$\varepsilon = \frac{1}{E}\sigma$$

The Linear Viscous Dash-pot

A material which responds like a viscous dash-pot; the dash-pot is a piston cylinder arrangement, filled with a viscous fluid, a strain is achieved by dragging the piston through the fluid.

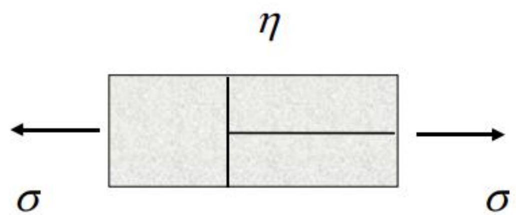


Figure 10.3.2: the linear dash-pot

$$\dot{\varepsilon} = \frac{1}{\eta}\sigma$$

where *η* is the viscosity of the material. This is the typical response of many fluids; the larger the stress, the faster the straining

The Linear Viscous Dash-pot

$$\dot{\varepsilon} = \frac{1}{\eta}\sigma$$

where η is the viscosity of the material. This is the typical response of many fluids; the larger the stress, the faster the straining. The strain due to a suddenly applied load η_o may be obtained by integrating the equation. Assuming zero initial strain, one has

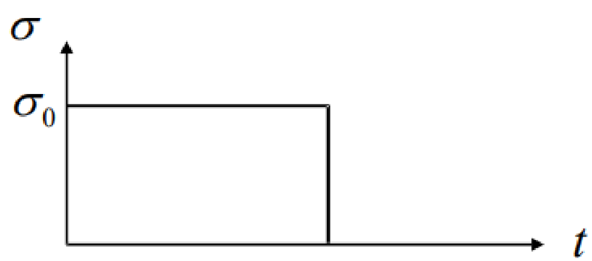
$$\varepsilon = \frac{\sigma_o}{\eta}t$$

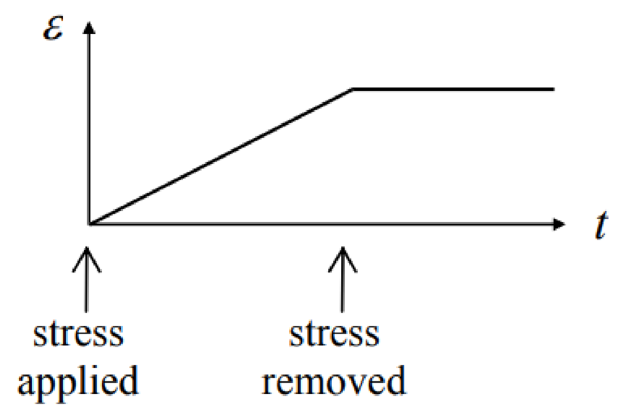
Creep and stress relaxation

- **Creep:** Change in deformation with time when the material is suddenly subjected to a dead load-constant stress.
- **Stress relaxation:** Decay of stress with time when the material is suddenly deformed to a given deformation-constant strain.

Creep-Recovery Response of the Dash-pot

- The strain is seen to increase linearly and without bound so long as the stress is applied, Note that there is no movement of the dash-pot at the onset of load; it takes time for the strain to build up. When the load is removed, there is no stress to move the piston back through the fluid, so that any strain built up is permanent.





Creep-Recovery Response of the Dash-pot

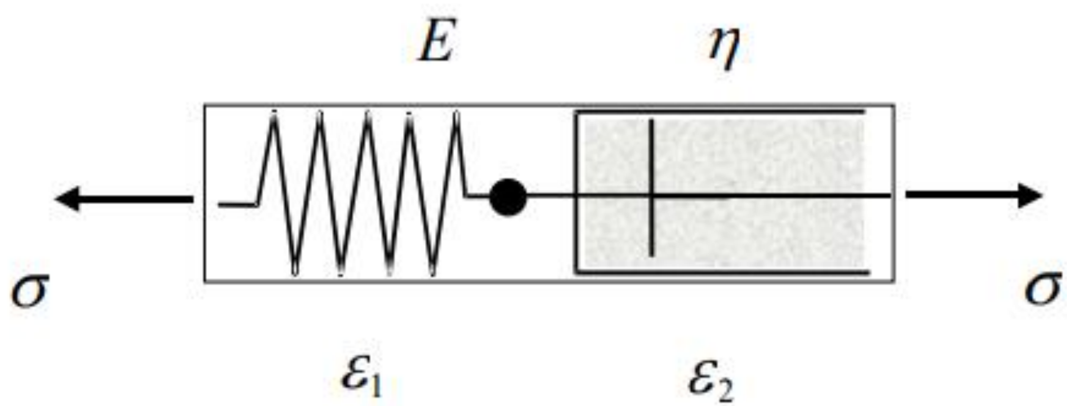
The relationship between the stress and strain during the creep-test may be expressed in the form

$$\varepsilon(t) = \sigma_o J(t), \quad J(t) = \frac{t}{\eta} \tag{10.3.4}$$

J here is called the **creep (compliance) function** ($J = 1/E$ for the elastic spring).

The Maxwell Model

- A spring and dash-pot in series are kept in Maxwell model. One can divide the total strain into one for the spring (ε_1) and one for the dash-pot (ε_2).



$$\sigma = \sigma_1 = \sigma_2$$

$$\epsilon = \epsilon_1 + \epsilon_2$$

- Equilibrium requires that the stress be the same in both elements. One thus has the following three equations -

$$\varepsilon_1 = \frac{1}{E}\sigma, \qquad \dot{\varepsilon}_2 = \frac{1}{\eta}\sigma, \qquad \varepsilon = \varepsilon_1 + \varepsilon_2$$

- To eliminate ε_1 and ε_2 , differentiate the first and third equations, and put the first and second into the third:

$$\boxed{\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon}} \qquad \textbf{Maxwell Model}$$

Creep-Recovery Response

- Physically, when the Maxwell model is subjected to a stress σ_0 , the spring will stretch immediately and the dash-pot will take time to react. Thus the initial strain is $\varepsilon(0) = \sigma_0 / E$. Using this as the initial condition

$$\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon}$$

Creep-Recovery Response

$$\dot{\varepsilon} = \frac{\sigma_o}{\eta} \quad \rightarrow \quad \varepsilon(t) = \frac{\sigma_o}{\eta} t + C$$

$$\rightarrow \quad \varepsilon(t) = \sigma_o \left(\frac{1}{\eta} t + \frac{1}{E} \right) \qquad ($$

The creep-response can again be expressed in terms of a creep compliance function

$$\varepsilon(t) = \sigma_o J(t) \quad \text{where} \quad J(t) = \frac{t}{\eta} + \frac{1}{E} \qquad ($$

Maxwell Model Characteristics

- Creep:
  - For constant stress, we get:

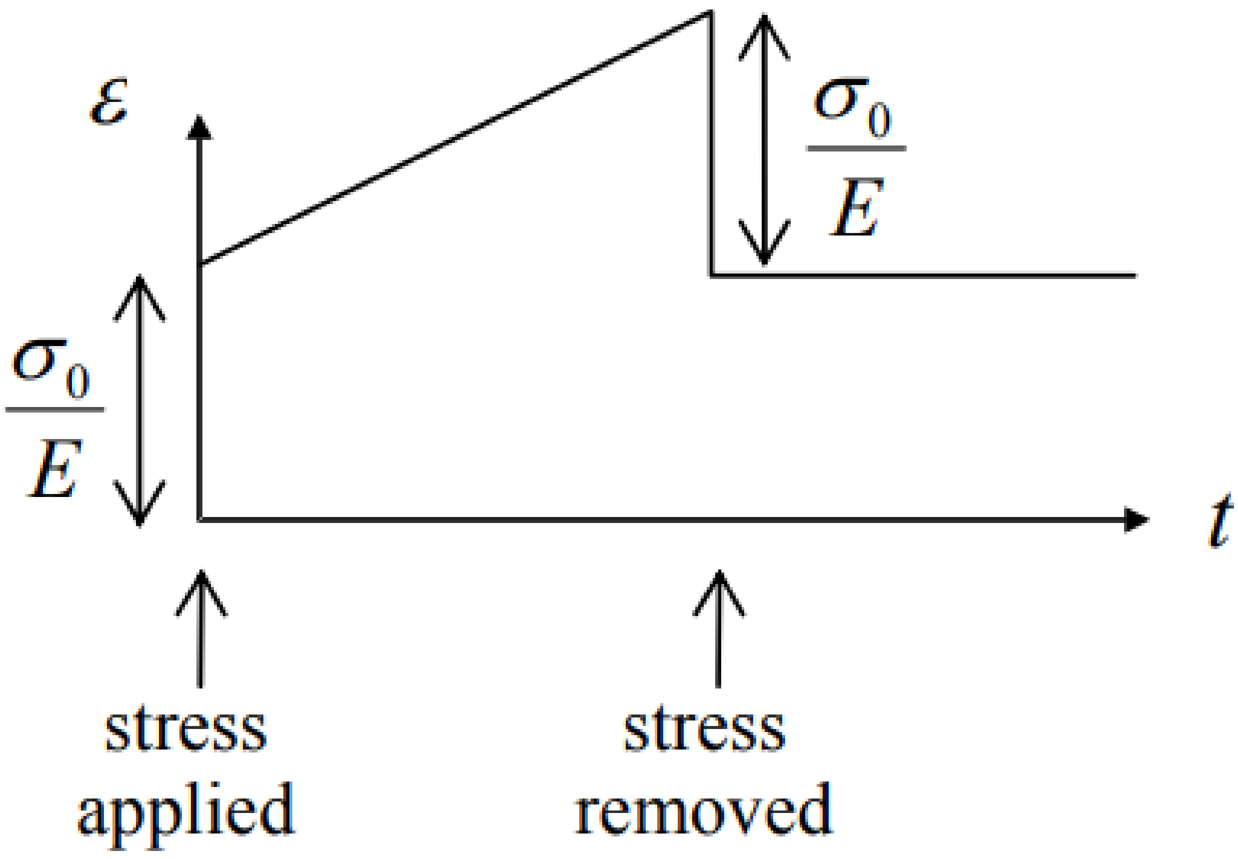
$$\dot{\varepsilon}=\underbrace{\frac{\dot{\sigma}}{E_s}}_{zero}+\frac{\sigma}{C_d}$$

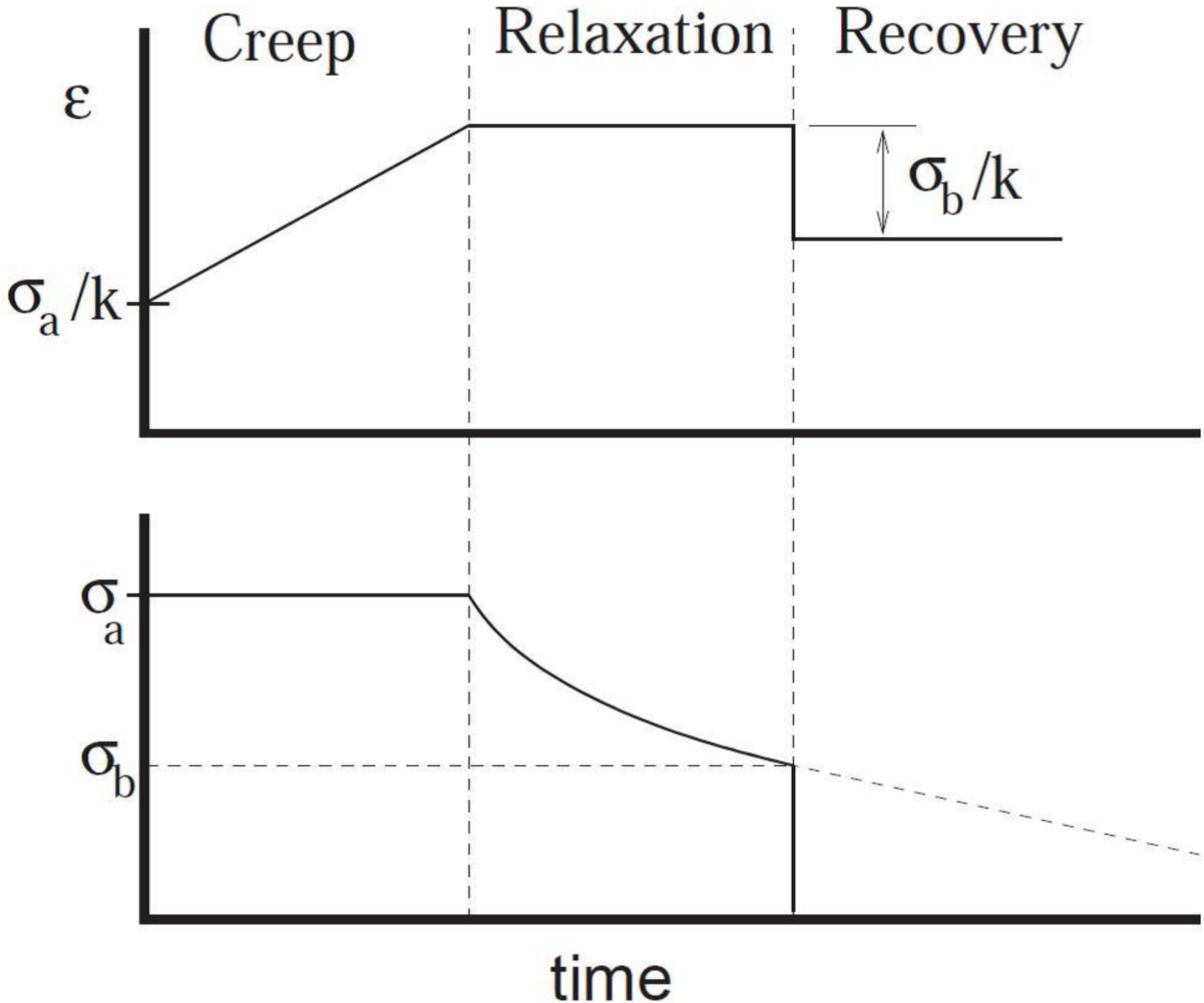
  - Which gives:

$$\varepsilon=\frac{\sigma}{C_d}t$$

- Which indicates that the strain will **grow to an unbound value** as time increases!

Creep-Recovery Response




Creep
Relaxation
Recovery
ε
σb /k
σa /k
σa
σb
time

Creep-Recovery Response

- When the load is removed, the spring again reacts immediately, but the dash-pot has no tendency to recover. Hence there is an immediate elastic recovery σ_0 /E, with the creep strain due to the dash-pot remaining.
- The Maxwell model predicts creep, but not of the ever-decreasing strain-rate type. There is no an elastic recovery, but there is the elastic response and a permanent strain.

Stress Relaxation

- In the stress relaxation test, the material is subjected to a constant strain ε_1 at t =0 . The Maxwell model then leads to

$$\sigma(t) = \varepsilon_o E(t) \quad \text{where} \quad E(t) = Ee^{-t/t_R}, \quad t_R = \frac{\eta}{E}$$

- Analogous to the creep function *J* for the creep test, E(t) is called the **relaxation modulus** function.
- The parameter t_R is called the **relaxation time** of the material and is a measure of the time taken for the stress to relax; the shorter the relaxation time, the more rapid the stress relaxation.

Relaxation time

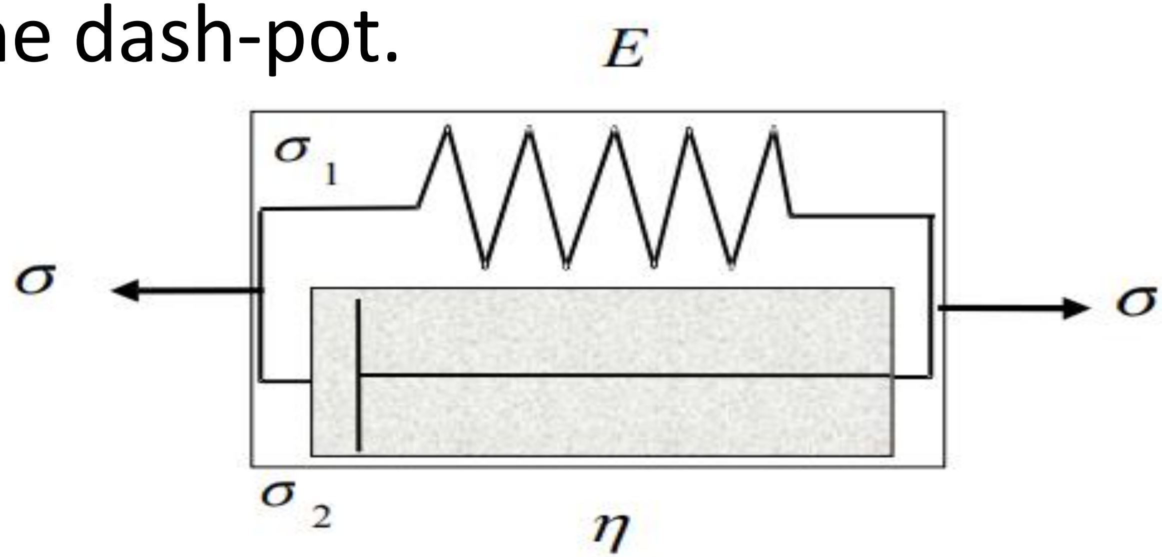
- **Relaxation time:** The rate of stress decay in-a material subjected to a sudden strain. It is the time required for the stress in the Maxwell model, representing stress relaxation behavior, to decay to (1/e) or approximately 37 percent of its original value.

Maxwell Model Characteristics

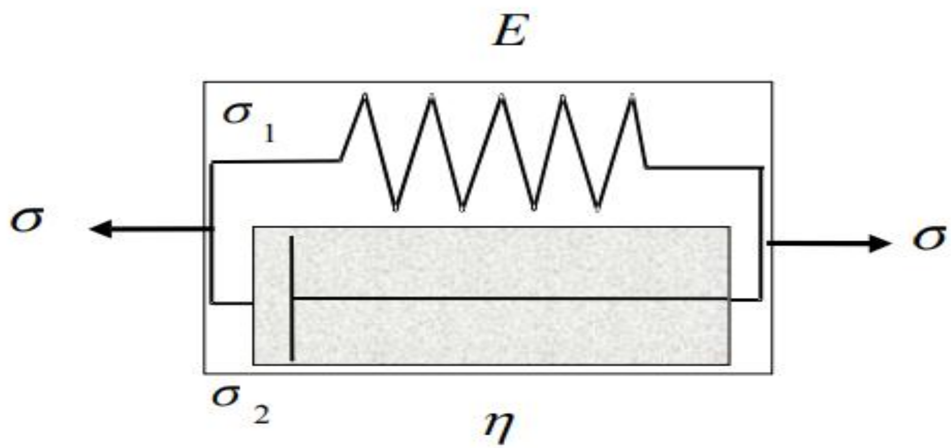
- Relaxation:
  - For constant strain, we get: $$0 = \frac{\dot{\sigma}}{E_s} + \frac{\sigma}{C_d}$$
  - Which gives: $$\sigma = \sigma_0 e^{-tE_s/C_d}$$
- Which means that the **stress will decrease** as time grows for the same strain

The Kelvin (Voigt) Model

- the Kelvin (or Voigt) model, which consists of a spring and dash-pot in parallel. It is assumed there is no bending in this type of parallel arrangement, so that the strain experienced by the spring is the same as that experienced by the dash-pot.



The Kelvin (Voigt) Model



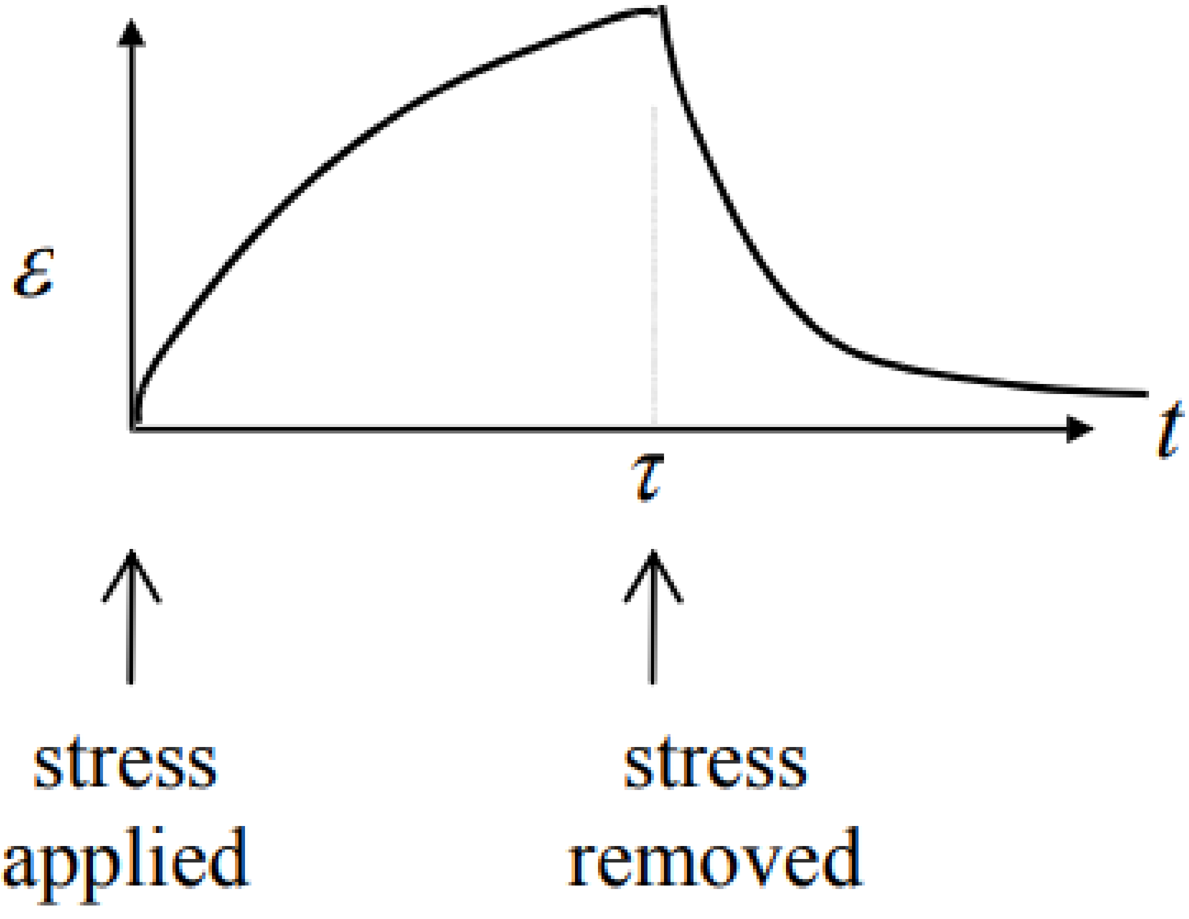
$$\varepsilon = \frac{1}{E}\sigma_1, \quad \dot{\varepsilon} = \frac{1}{\eta}\sigma_2, \quad \sigma = \sigma_1 + \sigma_2$$

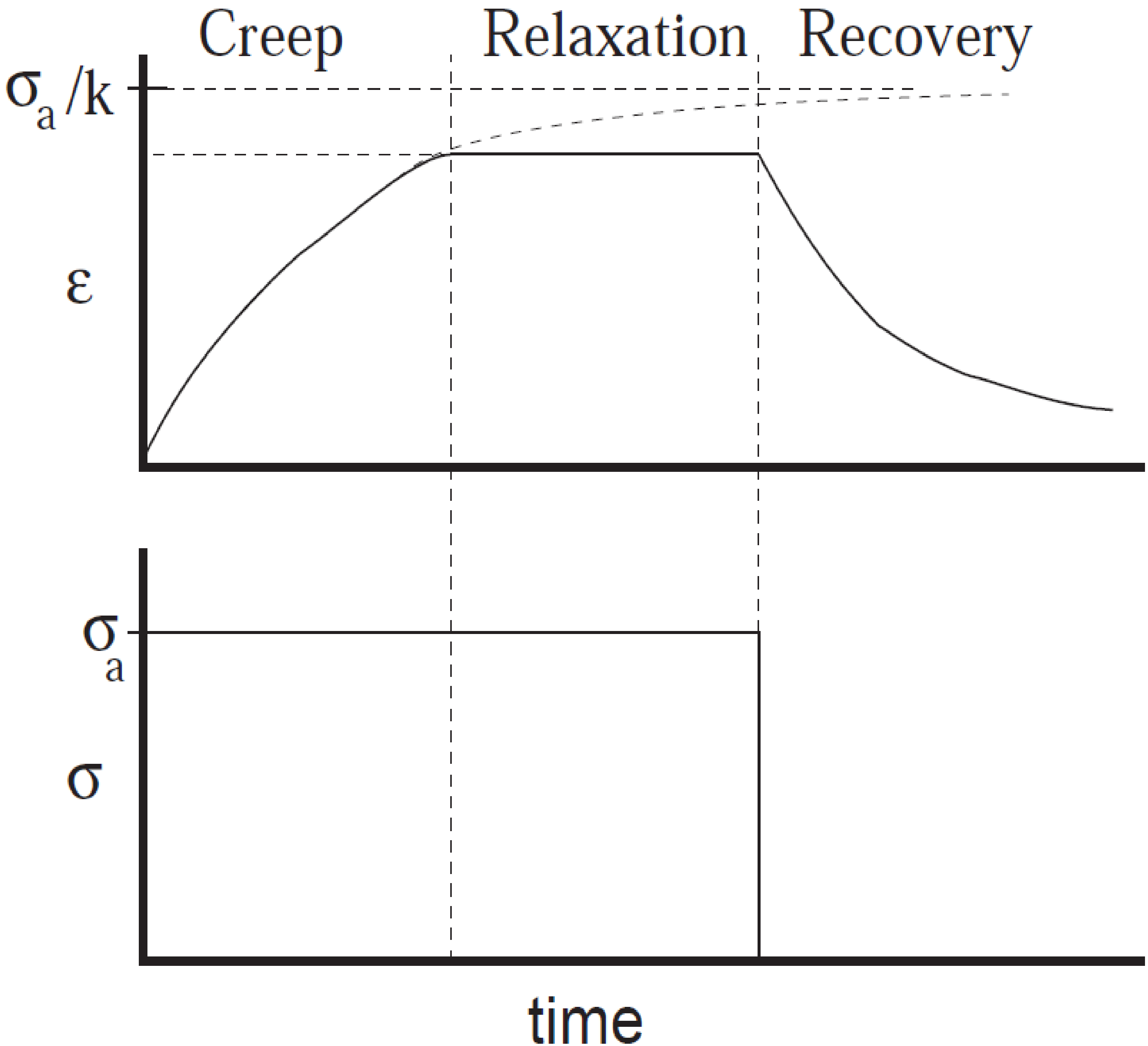
$$\boxed{\sigma = E\varepsilon + \eta\dot{\varepsilon}}$$ **Kelvin (Voigt) Model**

Creep-Recovery Response

- If a load σ_0 is applied suddenly to the Kelvin model, the spring will want to stretch, but is held back by the dash-pot, which cannot react immediately.
- Since the spring does not change length, the stress is initially taken up by the dash-pot. The creep curve thus starts with an initial slope σ_0 / η.

Creep-Recovery Response




Creep
Relaxation
Recovery
σ_a/k
ε
σ_a
σ
time

Creep-Recovery Response

- Some strain then occurs and so some of the stress is transferred from the dash-pot to the spring.
- The slope of the creep curve is now σ_2 /η , where σ_2 is the stress in the dashpot, with σ_2 ever-decreasing.
- In the limit when σ_2=0, the spring takes all the stress and thus the maximum strain is σ_0 / E .

Creep-Recovery Response

- Solving the first order non-homogeneous differential equation with the initial condition Ɛ(0)=0 gives

$$\varepsilon(t) = \frac{\sigma_o}{E}\left(1 - e^{-(E/\eta)t}\right)$$

- which agrees with the above physical reasoning; the creep compliance function is now

$$J(t) = \frac{1}{E}\left(1 - e^{-t/t_R}\right), \quad t_R = \frac{\eta}{E}$$

Creep-Recovery Response

- The parameter t_R in contrast to the relaxation time of the Maxwell model, is here called the **retardation time** of the material and is a measure of the time taken for the creep strain to accumulate; the shorter the retardation time, the more rapid the creep straining.
- When the Kelvin model is unloaded, the spring will want to contract but again the dash pot will hold it back. The spring will however eventually pull the dash-pot back to its original zero position given time and full recovery occurs.

Retardation time

- **Retardation time:** The rate at which the retarded elastic deformation takes place in a material creeping under dead load. It is the time required for the Kelvin model, representing creep behavior, to deform to (1-(1/e)) or about 63 percent of its total deformation.

Creep-Recovery Response

- Suppose the material is unloaded at time t=ζ. The constitutive law, with zero stress, reduces to

$$0 = E\varepsilon + \eta\dot{\varepsilon}\ .$$

$$\varepsilon(t) = Ce^{-(E/\eta)t}$$

Creep-Recovery Response

- where C is a constant of integration. The t here is measured from the point where "zero load" begins. If one wants to measure time from the onset of load, t must be replaced with t=ζ. the strain at

$$t = \tau \text{ is } \varepsilon(\tau) = (\sigma_0 / E)\left(1 - e^{-(E/\eta)\tau}\right).$$

Stress Relaxation

- Consider next a stress-relaxation test. Setting the strain to be a constant ε_0 , $\sigma = E\,\varepsilon_0$
- Thus the stress is taken up by the spring and is constant, so there is in fact no stress relaxation over time.
- Actually, in order that the Kelvin model undergoes an instantaneous strain of ε_0 , an infinite stress needs to be applied, since the dash-pot will not respond instantaneously to a finite stress.

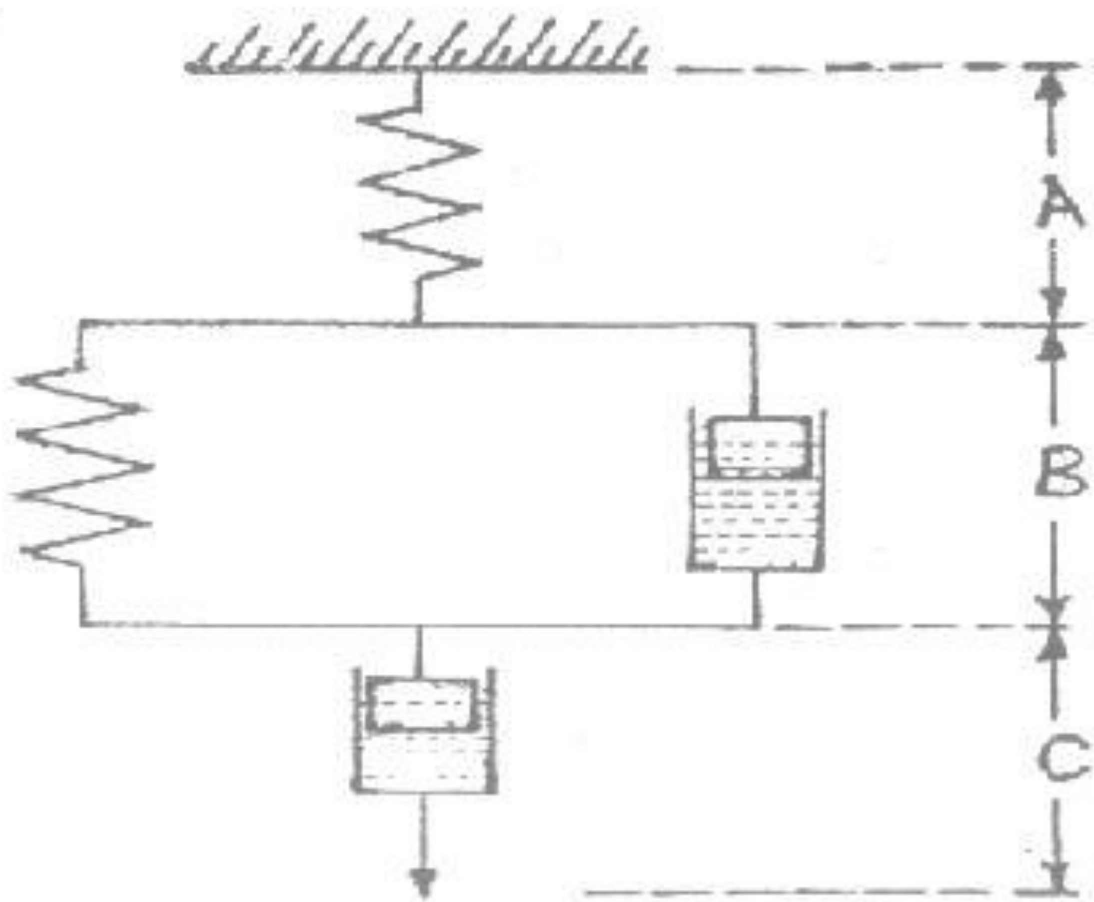
Creep relaxation summary

	Maxwell	**Kelvin-Voigt**
Creep	Bad	Good
Relaxation	Good	Bad
Recovery	Bad	Good

4-Element model (Burgers model)

- Burger’smodel-states that the behavior of agricultural materials under stress can be represented by a spring (representing elastic behaviour) in series with a dashpot and a combined spring and a dashpot (representing viscous behaviour) in parallel.

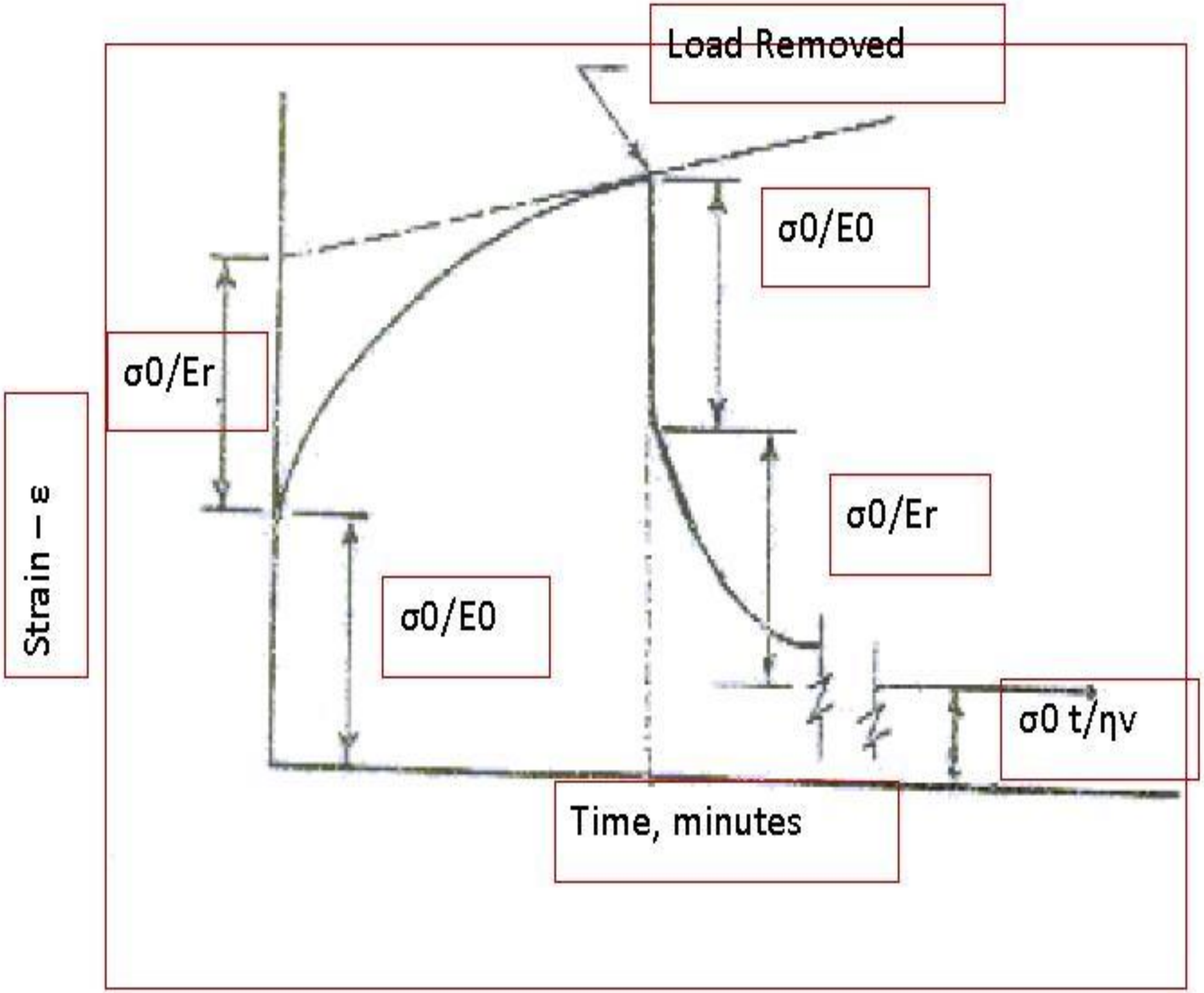
4-Element model (Burgers model)



Power law model

- The power law model with or without a yield term has been employed extensively to describe the flow behavior of viscous foods over wide ranges of shear rates where ‘K’ is the consistency coefficient (consistent index, Pa.s^n), and ‘n’ is the flow behavior index (dimensionless) is also known as the **Herschel-Bulkley model**.

$$\tau = \tau_0 + K\dot{\gamma}^n$$



Typical creep and recovery curve in a viscoelastic material exhibiting instantaneous elasticity, retarded elasticity and viscous flow.