

Dimensional & Model Analysis

Dimensional analysis: is a mathematical technique used in research work for designing and conducting model tests.

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Fundamental and derived quantities

Fundamental quantities: Length (L), mass (M), time (T)

Some time temperature (θ) is also used as a fundamental quantity.

Secondary quantities: are those which possess more than one fundamental dimension.

Example: Velocity (LT^{-1})
Acceleration (LT^{-2})
Density (ML^{-3})

Dimensional homogeneity

Dimensional homogeneity: means the dimensions of each terms in an equation on both sides are the same.

If the dimensions of each term on both sides of an equation are the same the equation is known as dimensionally homogeneous equation.

Example:

$$S = ut + \frac{1}{2}at^2$$

$$[S] = L$$

$$[ut] = [LT^{-1}T] = [L]$$

$$\left[\frac{1}{2}at^2 \right] = [LT^{-2}T^2] = [L]$$

It is a dimensionally homogeneous equation.

Methods of dimensional analysis

If the number of variables involved in a physical phenomenon are known, then the relation among the variables can be determined by the following two methods.

- Reyleigh's method
- Buckingham's Pi-theorem

Reyleigh's method

Let X is a variable, which depends on X_1 , X_2 and X_3 , then

$$X = f[X_1, X_2, X_3]$$

This can be written as $X = K X_1^a \cdot X_2^b \cdot X_3^c$

where K is a constant and a , b , c are arbitrary powers. Then according to Reyleigh's theorem a , b , c are obtained by comparing the powers of the fundamental dimensions on both sides. Thus the expression is obtained for dependent variables.

Limitation: This method becomes more laborious if the variables are more than the number of fundamental dimensions.

Buckingham's Pi-theorem

This states that if there are n variables (independent and dependent) in a physical phenomenon and if these variables contain m fundamental dimensions (M, L, T), then the variables are arranged into $(n-m)$ dimensionless groups, known as π – terms.

Each π – term contain $m+1$ variables, out of which m are repeating variables.

Method of selecting repeating variables

- The dependent variable should not be selected as repeating variable.
- The dependent variables should be chosen in such a way that one variable contains geometric property, other variable contains flow property and third variable contains fluid property.

Example:

Geometric properties: length, diameter, height

Flow properties: velocity, acceleration

Fluid properties: viscosity, density

- The repeating variable should not form a dimensionless group
- The repeating variables should have the same number of fundamental dimensions.
- No two repeating variables should have the same fundamental dimensions.

Model analysis

Model: is the small scale replica of the actual structure or machine. It is not necessary that models should be smaller than the prototypes (although in most of the cases it is), they may be larger than the prototypes.

Prototype: The actual structure or machine

Model analysis: the study of models of actual machine.

Advantages:

- The performance of the machine can be easily predicted, in advance.
- With the help of dimensional analysis, a relationship between the variables influencing a flow problem in terms of dimensional parameters is obtained. This relationship helps in conducting tests on the model.
- The merits of alternative designs can be predicted with the help of model testing. The most economical and safe design may be, finally, adopted.

Types of similarities between model and prototype

Three type of similarities must exist between the model and prototype.

Geometric Similarity:

The ratio of all corresponding linear dimension in the model and prototype are equal.

Let L_m = length of model

b_m = width of model

d_m = diameter of model

A_m = area of model

V_m = volume of model

L_p, b_p, d_p, A_p, V_p are corresponding values of the prototype.

$$\frac{L_p}{L_m} = \frac{b_p}{b_m} = \frac{d_p}{d_m} = L_r$$

$$\frac{A_p}{A_m} = L_r^2, \quad \frac{V_p}{V_m} = L_r^3$$

Kinematic Similarity: means the similarity of motion between model and prototype. Thus kinematic similarity is said to exist between the model and the prototype if the ratios of the velocity and acceleration at the corresponding points in the model and prototype are the same in magnitude; the directions also should be parallel.

$$\frac{V_{p_1}}{V_{m_1}} = \frac{V_{p_2}}{V_{m_2}} = V_r \qquad \frac{a_{p_1}}{a_{m_1}} = \frac{a_{p_2}}{a_{m_2}} = a_r$$

Dynamic Similarity: means the similarity of forces between model and prototype. Thus dynamic similarity is said to exist between the model and the prototype if the ratios of the forces acting at the corresponding points in the model and prototype are the same in magnitude; the directions also should be parallel.

$$\frac{F_{p_1}}{F_{m_1}} = \frac{F_{p_2}}{F_{m_2}} = \frac{F_{p_3}}{F_{m_3}} = F_r$$

Type of forces acting in the moving fluid

Inertial force: it is equal to the mass and acceleration of the moving fluid.

$$F_i = \rho AV^2$$

Viscous force: it is equal to the shear stress due to viscosity and surface area of the flow. It present in the flow problems where viscosity is having an important role to play.

$$F_v = \tau A = \mu \frac{du}{dy} A = \mu \frac{U}{d} A$$

Gravity force: product of mass and acceleration due to gravity.

$$F_g = \rho ALg$$

Pressure force: product of pressure intensity and flow area.

$$F_p = pA$$

Surface tension force: product of surface tension and the length of the surface of the flowing fluid.

$$F_s = \sigma d$$

Elastic force: product of elastic stress and area of the flow.

$$F_e = \text{Elastic stress} \times \text{Area} = KA$$

Dimensionless numbers

Reynolds number: $Re = \frac{\text{Inertial force}}{\text{Viscous force}} = \frac{V \rho d}{\mu}$

Froude's number: $Fr = \sqrt{\frac{\text{Inertial force}}{\text{gravity force}}} = \frac{V}{\sqrt{dg}}$

Euler's number: $E_u = \frac{\text{Pressure force}}{\text{Inertial force}} = \frac{p}{\rho V^2}$

Weber's number: $W_e = \frac{\text{Surface tension force}}{\text{Inertial force}} = \frac{\sigma}{\rho V^2 l}$

Mach's number: $M = \sqrt{\frac{\text{Inertial force}}{\text{Elastic force}}} = \frac{V}{\sqrt{K / \rho}} = \frac{V}{c}$

c is velocity of sound.

Dynamic similarity

- For the dynamic similarity between the model and the prototype the ratio of the corresponding forces acting at the corresponding points in the model and prototype should be equal.
- It means for dynamic similarity between the model and prototype, the dimensionless numbers should be same for model and prototype.
- It is quite difficult to satisfy the condition that all the dimensionless numbers are the same for the model and prototype. Therefore model are designed on the basis of equating the dimensionless number which dominate the phenomenon.

Following are the dynamic similarity laws:

1. Reynolds model law
2. Froude model law
3. Euler model law
4. Weber model law
5. Mach model law

Reynolds model law: (Pipe flow, sub-marines, aeroplane etc)

$$[\text{Re}]_m = [\text{Re}]_p \Rightarrow \frac{V_m \rho_m d_m}{\mu_m} = \frac{V_p \rho_p d_p}{\mu_p}$$

Froude's model law: (Free-surface flow, jet from orifice or nozzle etc)

$$[\text{Fr}]_m = [\text{Fr}]_p \Rightarrow \frac{V_m}{\sqrt{g_m L_m}} = \frac{V_p}{\sqrt{g_p L_p}}$$

Euler's model law: (pressure force is a dominant force)

$$[E_u]_m = [E_u]_p \Rightarrow \frac{V_m}{\sqrt{p_m / \rho_m}} = \frac{V_p}{\sqrt{p_p / \rho_p}}$$

Weber model law: (surface tension is a dominant force)

$$[W_e]_m = [W_e]_p \Rightarrow \frac{V_m}{\sqrt{\sigma_m / \rho_m L_m}} = \frac{V_p}{\sqrt{\sigma_p / \rho_p L_p}}$$

Mach model law: (velocity of flow is comparable to the velocity of sound; compressible flow)

$$[M]_m = [M]_p \Rightarrow \frac{V_m}{\sqrt{K_m / \rho_m}} = \frac{V_p}{\sqrt{K_p / \rho_p}}$$

Classification of model

- **Undistorted models:** are those models which are geometrically similar to their prototype. In other words the scale ratio for the linear dimensions of the model and its prototype are the same.
- **Distorted models:** are those models which are geometrically not similar to its prototype. In other words the scale ratio for the linear dimensions of the model and its prototype are not same.

For example river:

If the horizontal and vertical scale ratios for the model and the prototype are same then it is undistorted model. In this case the depth of the water in the model becomes very small which may not be measured accurately.

Thus for cases distorted model is useful.